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On the sine-Gordon—Thirring equivalence in the presence of a boundary

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Abstract

In this paper, the relationship between the sine-Gordon model with an integrable boundary condition and the Thirring model with boundary is discussed and the reflection R-matrix for the massive Thirring model, which is related to the physical boundary parameters of the sine-Gordon model, is given. The relationship between the the boundary parameters and the two formal parameters appearing in the work of Ghoshal and Zamolodchikov is discussed.

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Introduction

Recently, much attention has been paid to the study of integrable field theory with boundary [1][2][3][4][5][6][7]. The study of such field theories is not only intrinsically interesting but also provides a better understanding of boundary related phenomena in statistical physics and condensed matter [8][9]. Probably the most famous physical example of a boundary integrable model is the Kondo problem, where the 1+1 dimensional field theory is an effective field theory of s-wave scattering of electrons off a magnetic spin impurity. Such an impurity problem, in which one concentrates on s-wave scattering from some isolated object at the origin, generically provides interesting 1+1 dimensional boundary field theory. There are also 1+1 dimensional boundary quantum integrable systems of experimental relevance, such as Luttinger liquid (Thirring) model for the edge states of electrons in the fractional quantum Hall effect [9] and the boundary sine-Gordon model may provide an accurate description of conductance through a point contact [10].

An integrable field theory possesses an infinite set of independent, commuting integrals of motion. In the 'bulk theory' these integrals of motion follow from an infinite number of conserved currents. However, when the theory is restricted to a half-line (or to an interval) the existence of conserved currents on the whole-line does not guarantee integrability unless special boundary conditions are specified. The integrable boundary conditions under which a theory preserves its integrability can be obtained in various ways: by the boundary action functional, via a perturbed conformal boundary condition [11][12] or through a Lax pair approach [3].

An important characteristic of an integrable field theory on the whole-line is its factorizable S-matrix. In the bulk theory, such an S-matrix is required to satisfy the Yang-Baxter equation (or 'factorizability condition'), in addition to the standard requirements of unitarity and crossing. These equations have much restrictive power, determining the S-matrix up to the so-called 'CDD ambiguity' [13][14][15]. For an integrable field theory with a boundary, particles cannot escape beyond the boundary and therefore reflect from it. The assumption of factorisibility for a theory defined on a half-line requires the factorizable S-matrix describing the scattering of particles in the bulk far from the boundary to be compatible with the matrix describing the reflection from the boundary (the boundary reflection matrix often denoted R or K). S and R are required to satisfy an appropriate generalization of the Yang-Baxter equation (the Boundary Yang-Baxter equation [1]), and also generalizations of unitarity and crossing (the Boundary Cross-Unitarity equation [1][16]). There is also a modified version of the bootstrap equations (the Boundary Bootstrap [17][18]) which ensures compatibility between the R, S descriptions of particles and their bound states.

A particularly interesting model of massive boundary field theory is the sine-Gordon model on a half line. It exhibits relationships with the theory of Jack Symmetric functions [10], and has applications to dissipative quantum mechanics [19] and impurity problems in a one-dimensional strongly correlated electron gas [20]. Including the integrable boundary condition, its action can be written [1]

$$\begin{aligned} S = & \int_{-\infty}^{+\infty} dt \int_{-\infty}^{0} dx \left[\tfrac{1}{2}(\partial_x\phi)^2 + \tfrac{1}{2}(\partial_t\phi)^2 - \tfrac{m_0^2}{\beta^2}\cos\beta\phi - \tfrac{m_0^2}{\beta^2}\right] \\ & -h\int_{-\infty}^{+\infty} dt \left(\cos\tfrac{\beta(\phi-\phi_0)}{2}\Big|_{x=0} - \cos\tfrac{\beta\phi_0}{2}\right), \end{aligned} \tag{1}$$

where $\phi(x,t)$ is a real scalar field, β is a dimensionless coupling constant and m_0 is the mass parameter, the constant term is added so that the energy is zero when $\phi = 0$. Its integrable boundary condition reads

$$\partial_x\phi = \frac{\beta}{2}h\sin\frac{\beta}{2}(\phi-\phi_0) \qquad \text{at} \quad x=0, \tag{2}$$

and it tends to a free boundary condition (or fixed boundary condition), when $h \to 0$ (or $h \to \infty$). The R matrix for this model (modulo the 'CDD ambiguity') was obtained by explicitly solving the boundary Yang-Baxter equation and the crossing-unitarity equations [1]. This general solution to the boundary Y-B equation and the C-U condition depends on two formal parameters. However the relationship between these formal parameters and the physical parameters (h and ϕ_0) related to the boundary term in the action were not given. There are also other methods for obtaining the R matrix, such as via the Bethe Ansatz [16][6] or from perturbed conformal field theory [22]. However, until now only the R-matrices for special boundary conditions, such as the free boundary condition, the fixed boundary condition and the free fermion point, were given. The study of how the R-matrix is related to the physical boundary parameters for the sine-Gordon model with a general integrable boundary condition remains to be undertaken. In this paper, we intend to improve this situation by giving such a R-matrix, making use of the well-known relationship between the Thirring Model and the sine-Gordon model [23][24].

One can construct a transformation to fermionize the bulk sine-Gordon model into the bulk massive Thirring model [24] in terms of the non-local transformation recently used in a discussion of the non-local currents of the sine-Gordon model [25][26][27][28]. However, there are several kinds of integrable boundary conditions for the Thirring Model (which is therefore rather different to the sine-Gordon model in which there is only one class of integrable boundary condition). For example, there are $SU(2)$ invariant or $U(1)$ invariant integrable boundary conditions [29][30][6]; the richer boundary structure in the Thirring model matches the situation in the bulk case [31][32]. If we wish, we can choose a boundary condition equivalent to a boundary condition of the sine-Gordon model but

which is linear in terms of the Thirring field. In the third section, we will give the R-matrix of the sine-Gordon model obtained using a special integrable boundary condition for the massive Thirring model.

Relation between sine-Gordon model and Thirring model with boundary

In order to fermionize the sine-Gordon model, the following non-local transformation may be introduced:

$$\left\{ \begin{array}{ll} \rho(x,t) & = \frac{1}{2}\left(\phi(x,t) + \int_{-\infty}^{x} \partial_t\phi(y,t)dy\right) \\ \bar{\rho}(x,t) & = \frac{1}{2}\left(\phi(x,t) - \int_{-\infty}^{x} \partial_t\phi(y,t)dy\right). \end{array} \right. \tag{3}$$

Using this transformation, we define new fields by

$$\begin{array}{llll} \psi_1(x) & = A : e^{-ia\rho(x)-ib\bar{\rho}(x)} : & \psi_1^\dagger(x) & = A : e^{ia\rho(x)+ib\bar{\rho}(x)} : \\ \psi_2(x) & = -iA : e^{ib\rho(x)+ia\bar{\rho}(x)} : & \psi_2^\dagger(x) & = iA : e^{-ib\rho(x)-ia\bar{\rho}(x)} : \end{array} \tag{4}$$

where $a = \frac{1}{2}(\beta + \frac{4\pi}{\beta})$, $\quad b = \frac{1}{2}(\beta - \frac{4\pi}{\beta})$, and A is a constant with the dimension $[M]^{\frac{1}{2}}$. It is easy to compute the anticommutators for the fields $\psi_i(x)$

$$\{\psi_i(x,t), \psi_j^\dagger(y,t)\} = \delta_{ij}\delta(x-y) \tag{5}$$

by using the canonical commutation relations for the field $\phi(x,t)$ and its canonical conjugate $\partial_t\phi(y,t)$

$$[\phi(x,t), \partial_t\phi(y,t)] = i\delta(x-y), \tag{6}$$

and the standard relation:

$$e^A e^B = e^{[A,B]} e^B e^A, \tag{7}$$

valid when $[A,B]$ commutes with both A and B.

Using the above transformation, one can obtain the Lagrangian of the bulk Thirring Model as follows [23][24]

$$\mathcal{L}_{\mathrm{T_{bulk}}} = \bar{\psi} i\gamma^\mu \partial_\mu \psi - \frac{g}{2} j^\mu j_\mu - M\bar{\psi}\psi, \tag{8}$$

where $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and $\bar{\psi} = \psi^\dagger \gamma^0 = (\psi_2^\dagger, \psi_1^\dagger)$ with

$$\gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the current components and the various parameters defined by:

$$\left\{\begin{array}{l} j^\mu(x) = \lim_{y\to x} \frac{1}{2}\left(\bar\psi(x)\gamma^\mu\psi(y) + \bar\psi(y)\gamma^\mu\psi(x)\right) = -\frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi(x) \\ \frac{4\pi}{\beta^2} = 1 + g/\pi \\ \frac{m_0^2}{\beta^2}\cos\beta\phi = ZM\bar\psi\psi \end{array}\right. , \tag{9}$$

here $\epsilon^{01} = 1$ and $Z = \frac{m_0^2}{2M\beta^2 A^2}$. From the first of eqs(9)[4], we know that $j^0(x) = 0$ at $x = 0$ corresponds to the free boundary condition of the sine-Gordon model, and $j^1(x) = 0$ at $x = 0$ corresponds to $\partial_t\phi(0,t) = 0$ or $\phi(0,t) = \phi_0$ which is the fixed boundary condition. Explicitly,

$$j^0(x,t)\Big|_{x=0} = \frac{1}{2}(\psi_1^\dagger(0)\psi_1(0^-) + \psi_2^\dagger(0)\psi_2(0^-) + \psi_1^\dagger(0^-)\psi_1(0) + \psi_2^\dagger(0^-)\psi_2(0)) = 0 \tag{10}$$

or

$$j^1(x,t)\Big|_{x=0} = \frac{1}{2}(\psi_1^\dagger(0)\psi_1(0^-) - \psi_2^\dagger(0)\psi_2(0^-) + \psi_1^\dagger(0^-)\psi_1(0) - \psi_2^\dagger(0^-)\psi_2(0)) = 0, \tag{11}$$

which imply the components $\psi_1(x)$ and $\psi_2(x)$ must be dependent at $x = 0$. Note also, if

$$\left\{\begin{array}{ll} \psi_1^\dagger(0) & = \mu\psi_2(0) \\ \psi_2^\dagger(0) & = \mu\psi_1(0), \end{array}\right. \tag{12}$$

where μ is a phase commuting with the components of ψ, then $j^0(0) = 0$ corresponding to the free boundary condition. On the other hand, if

$$\left\{\begin{array}{ll} \psi_1(0) & = e^{-i\beta\phi_0}\psi_2(0) \\ \psi_1^\dagger(0) & = e^{i\beta\phi_0}\psi_2^\dagger(0), \end{array}\right. \tag{13}$$

then $j^1(0) = 0$ which corresponds to the fixed boundary condition

$$\phi = \phi_0 + \frac{2\pi n}{\beta}.$$

The Thirring model restricted to the half-line with a general boundary condition may be thought of as a perturbation of the action containing the free condition:

$$\mathcal{L}_{\mathrm{T_b}} = \mathcal{L}_{\mathrm{T_{free}}} + \mathcal{L}_{\mathrm{T_{boundary}}}, \tag{14}$$

[4]This form of j^μ was taken in [33] to avoid the singularity

where, the action with free boundary condition reads,

$$\mathcal{L}_{\mathrm{T}_{\mathrm{free}}} = \mathcal{L}_{\mathrm{T}_{\mathrm{bulk}}} + i\mu\left(\psi_2(0)\psi_1(x) - \psi_1(0)\psi_2(x)\right)\delta(x) + i\mu^\dagger\left(\psi_2^\dagger(x)\psi_1^\dagger(0) - \psi_1^\dagger(x)\psi_2^\dagger(0)\right)\delta(x). \tag{15}$$

(From this action, one can easily obtain the free boundary condition, eq(12).)

Next, we want to find the boundary Lagrangian density corresponding to the general sine-Gordon boundary term. Noting,

$$:\cos\beta\left(\frac{\phi-\phi_0}{2}\right): = \frac{1}{2}:e^{i\frac{\beta\phi}{2}}:e^{-i\frac{\beta\phi_0}{2}} + \frac{1}{2}:e^{-i\frac{\beta\phi}{2}}:e^{i\frac{\beta\phi_0}{2}}, \tag{16}$$

we introduce the zero mode operators (or 'boundary field operators'), $b_\pm$:

$$\begin{aligned} b_- &= \tfrac{1}{4A}\exp\left[-\tfrac{i\pi}{2}(1+\tfrac{4}{\beta}\textstyle\int_{-\infty}^0\dot\phi(\xi,t)d\xi)\right] \\ b_+ &= \tfrac{1}{4A}\exp\left[\tfrac{i\pi}{2}(1+\tfrac{4}{\beta}\textstyle\int_{-\infty}^0\dot\phi(\xi,t)d\xi)\right], \end{aligned} \tag{17}$$

and use them to construct the required term. Since one can check that $b_\pm$ anticommutate with the components of ψ and $\psi^\dagger$, eq(16) becomes

$$:\cos\beta\left(\frac{\phi-\phi_0}{2}\right): = (\psi_1^\dagger b_- + \psi_2 b_+)e^{-i\frac{\beta\phi_0}{2}} - (\psi_1 b_+ + \psi_2^\dagger b_-)e^{i\frac{\beta\phi_0}{2}} \tag{18}$$

and the boundary Lagrangian density is

$$\begin{aligned} \mathcal{L}_{\mathrm{T}_{\mathrm{boundary}}} = \; & h\left((\psi_1^\dagger b_- - b_+\psi_2)e^{-i\frac{\beta\phi_0}{2}} + (b_+\psi_1 - \psi_2^\dagger b_-)e^{i\frac{\beta\phi_0}{2}} - \cos\beta\tfrac{\phi_0}{2}\right)\delta(x) \\ & + ihb_+\partial_t b_-\delta(x). \end{aligned} \tag{19}$$

Varying the action (14) with respect to the fermion fields and boundary field operator, we get the boundary conditions:

$$\begin{cases} (\psi_1^\dagger - \mu\psi_2)e^{-i\beta\phi_0} - (\psi_2^\dagger - \mu\psi_1) = 0 \\ \partial_t(\psi_1 - \mu^\dagger\psi_2^\dagger) + h(\psi_2 e^{-i\beta\phi_0} - \psi_1) = 0 \\ \partial_t(\psi_1^\dagger - \mu\psi_2) - h(\psi_1^\dagger - \psi_2^\dagger e^{i\beta\phi_0}) = 0. \end{cases} \tag{20}$$

It is worth noting that while the boundary condition (2) is the unique form of integrable boundary condition for the sine-Gordon model, at least on the assumption its boundary potential $\mathcal{B}(\phi)$ only depends on ϕ, the boundary condition (20) is not the only integrable boundary condition for the massive Thirring model even if its boundary potential depends only on ψ_i (and not its x-derivatives) at $x=0$.

R matrix related to the boundary parameters of sine-Gordon

The bulk theory (8) contains two types of fermion, a 'soliton' and an 'antisoliton', each of mass M. The corresponding particle creation operators $A_-^\dagger(\theta)$ and $A_+^\dagger(\theta)$ can be defined through the decomposition:

$$\begin{cases} \psi_1(x) = -i\sqrt{\frac{M}{4\pi}}\int d\theta e^{\theta/2}(A_+(\theta)e^{iMx\sinh\theta - iMt\cosh\theta} - A_-^\dagger(\theta)e^{-iMx\sinh\theta+iMt\cosh\theta}) \\ \psi_1^\dagger(x) = -i\sqrt{\frac{M}{4\pi}}\int d\theta e^{\theta/2}(A_-(\theta)e^{iMx\sinh\theta - iMt\cosh\theta} - A_+^\dagger(\theta)e^{-iMx\sinh\theta+iMt\cosh\theta}) \\ \psi_2(x) = -\sqrt{\frac{M}{4\pi}}\int d\theta e^{-\theta/2}(A_+(\theta)e^{iMx\sinh\theta - iMt\cosh\theta} + A_-^\dagger(\theta)e^{-iMx\sinh\theta+iMt\cosh\theta}) \\ \psi_2^\dagger(x) = -\sqrt{\frac{M}{4\pi}}\int d\theta e^{-\theta/2}(A_-(\theta)e^{iMx\sinh\theta - iMt\cosh\theta} + A_+^\dagger(\theta)e^{-iMx\sinh\theta+iMt\cosh\theta}), \end{cases} \tag{21}$$

where θ is the usual rapidity variable, and momentum and energy are given by

$$P = M\sinh\theta, \qquad E = M\cosh\theta,$$

M is the mass of soliton or antisoliton. The exact relation between M and m_0 is [36]

$$M = \mathcal{K}(\beta)m_0^{8\pi/\beta^2\lambda}\Lambda^{-1/\lambda}, \tag{22}$$

where Λ is an ultraviolet cutoff, $\lambda = \frac{8\pi}{\beta^2} - 1$, and

$$\mathcal{K}(\beta) = \frac{2\Gamma(\frac{1}{2\lambda})}{\sqrt{\pi}\Gamma(\frac{\lambda+1}{2\lambda})}\left(\frac{(\lambda+1)\Gamma(\frac{\lambda}{\lambda+1})}{16\Gamma(\frac{1}{1+\lambda})}\right)^{\frac{\lambda+1}{2\lambda}}.$$

As $\beta \to 0$, $\lambda \to \infty$, $\mathcal{K}(\beta) \to \frac{8}{\beta^2}$, one obtains the well known classical expression:

$$M = \frac{8m_0}{\beta^2}.$$

Using the anti-commutation relations of ψ_i, one calculates the anti-commutators for A's to be:

$$\begin{aligned} \{A_\pm(\theta), A_\pm^\dagger(\theta')\} &= \delta(\theta-\theta') \\ \{A_\pm(\theta), A_\mp^\dagger(\theta')\} &= 0. \end{aligned} \tag{23}$$

Substituting eq(21) into the boundary conditions (20), we find the following relations

$$\begin{aligned} &\left(iM\cosh\theta e^{\theta/2} + ihe^{-\theta/2}e^{-i\beta\phi_0} - he^{\theta/2}\right)A_-^\dagger(\theta)\,B + \mu^\dagger M\cosh\theta e^{-\theta/2}A_+^\dagger(\theta)\,B \\ &+\left(iM\cosh\theta e^{-\theta/2} + ihe^{\theta/2}e^{-i\beta\phi_0} - he^{-\theta/2}\right)A_-^\dagger(-\theta)\,B \\ &+\mu^\dagger M\cosh\theta e^{\theta/2}A_+^\dagger(-\theta)\,B = 0 \end{aligned} \tag{24}$$

and

$$
\begin{aligned}
&\left(iM\cosh\theta e^{\theta/2} + ihe^{-\theta/2}e^{i\beta\phi_0} - he^{\theta/2}\right) A_+^\dagger(\theta)\, B + \mu M\cosh\theta e^{-\theta/2} A_-^\dagger(\theta)\, B \\
&+ \left(iM\cosh\theta e^{-\theta/2} + ihe^{\theta/2}e^{i\beta\phi_0} - he^{-\theta/2}\right) A_+^\dagger(-\theta)\, B \\
&+\mu M\cosh\theta e^{\theta/2} A_-^\dagger(-\theta)\, B = 0,
\end{aligned}
\tag{25}
$$

where B represents any boundary state.

Since B represents an arbitrary boundary state, eqs(24,25) imply expressions for $A_\pm^\dagger(\theta)$ in terms of $A_\pm^\dagger(-\theta)$ which may be conveniently written in the form

$$
\begin{pmatrix} A_+^\dagger(\theta) \\ A_-^\dagger(\theta) \end{pmatrix} B = R(\theta) \begin{pmatrix} A_+^\dagger(-\theta) \\ A_-^\dagger(-\theta) \end{pmatrix} B \tag{26}
$$

with the R-matrix parametrised by

$$
R(\theta) = \begin{pmatrix} P^+(\theta) & Q^+(\theta) \\ Q^-(\theta) & P^-(\theta) \end{pmatrix},
$$

where

$$
\begin{cases}
P^+(\theta) & = \left(M\cosh\theta + h\cosh(\theta + i\beta\phi_0)\right)/P(\theta) \\
P^-(\theta) & = \left(M\cosh\theta + h\cosh(\theta - i\beta\phi_0)\right)/P(\theta) \\
Q^+(\theta) & = -i\mu M\cosh\theta\sinh\theta/P(\theta) \\
Q^-(\theta) & = -i\mu^\dagger M\cosh\theta\sinh 2\theta/P(\theta) \\
P(\theta) & = -M\cosh^2\theta - h(\cos\beta\phi_0 + i\sinh\theta).
\end{cases}
\tag{27}
$$

Returning to eq(21), we find the anticommutators eq(5) and eq(23) do not change under the following transformations

$$
\begin{cases}
A_+(\theta) & \to f(\theta)A_+(\theta) \\
A_-(\theta) & \to g(\theta)A_-(\theta)
\end{cases}
\tag{28}
$$

$$
\begin{cases}
A_+^\dagger(\theta) & \to f^\dagger(\theta)A_+^\dagger(\theta) \\
A_-^\dagger(\theta) & \to g^\dagger(\theta)A_-^\dagger(\theta),
\end{cases}
\tag{29}
$$

where $f(\theta)$, $g(\theta)$ are some arbitrary functions of θ which satisfy the following relations

$$
f(\theta)f^\dagger(\theta) = 1, \qquad g(\theta)g^\dagger(\theta) = 1 \tag{30}
$$

In other words, the R-matrix has an additional degree of freedom which is similar to the 'CDD ambiguity '. Under the above transformations, the R-matrix becomes

$$R(\theta) \to R^{'} = \begin{pmatrix} f(\theta) & 0 \\ 0 & g(\theta) \end{pmatrix} R(\theta) \begin{pmatrix} f^{\dagger}(-\theta) & 0 \\ 0 & g^{\dagger}(-\theta) \end{pmatrix} \tag{31}$$

or

$$\begin{cases} P^{'+}(\theta) = f(\theta) f^{\dagger}(-\theta) P^{\dagger}(\theta) \\ P^{'-}(\theta) = g(\theta) g^{\dagger}(-\theta) P^{-}(\theta) \\ Q^{'+}(\theta) = f(\theta) g^{\dagger}(-\theta) Q^{\dagger}(\theta) \\ Q^{'-}(\theta) = g(\theta) f^{\dagger}(-\theta) Q^{-}(\theta). \end{cases}$$

It is obvious that $P^{\pm}(\theta)$ do not change under the transformations (28) and (29) when $f(\theta)$ and $g(\theta)$ are both even functions of θ. Moreover, $P^{\pm}(\theta)P^{\pm}(-\theta)$ and $Q^{\pm}(\theta)Q^{\mp}(-\theta)$ are invariant. If $g(\theta) = f(\theta)$, then $R^{'}(\theta) = f(\theta)f^{\dagger}(-\theta)R(\theta)$, it is just the 'CDD ambiguity'. Now we should see if the $R(\theta)$ satisfies the Boundary Yang-Baxter equation , Unitarity condition and the Crossing Symmetry. It is easy to check that the $R(\theta)$ satisfies the Boundary Unitarity condition

$$R_a^c(\theta) R_c^b(-\theta) = \delta_a^b, \tag{32}$$

we found the $R(\theta)$ satisfies the following Boundary Yang-Baxter equation

$$R_{a_2}^{c_2}(\lambda\theta_2) S_{a_1 c_2}^{c_1 d_2}(\theta_1+\theta_2) R_{c_1}^{d_1}(\lambda\theta_1) S_{d_2 d_1}^{b_2 b_1}(\theta_1-\theta_2) = S_{a_1 a_2}^{d_1 d_2}(\theta_1-\theta_2) R_{d_1}^{c_1}(\lambda\theta_1) S_{d_2 c_1}^{c_2 b_1}(\theta_1+\theta_2) R_{c_2}^{b_1}(\lambda\theta_2) \tag{33}$$

where S is the S matrix of Sine-Gordon model or Thirring model:

$$\begin{cases} S_{11}^{11}(\theta) = S_{22}^{22}(\theta) = \sin(\lambda(\pi + i\theta))\rho(\theta) \\ S_{12}^{12}(\theta) = S_{21}^{21}(\theta) = -\sin(i\lambda\theta)\rho(\theta) \\ S_{12}^{21}(\theta) = S_{21}^{12}(\theta) = \sin(\lambda(\pi)\rho(\theta), \end{cases} \tag{34}$$

where $\rho(\theta)$ is

$$\rho(\theta) = -\tfrac{1}{\pi}\Gamma(\lambda)\Gamma(1+\tfrac{i\lambda\theta}{\pi})\Gamma(1-\lambda-\tfrac{i\lambda\theta}{\pi}) \times \Pi_{l=1}^{\infty}\tfrac{F_l(-i\theta)F_l(\pi+i\theta)}{F_l(0)F_l(\pi)}$$
$$F_l(x) = \tfrac{\Gamma(2l\lambda-\lambda x/\pi)\Gamma(1+2l\lambda-\lambda x/\pi)}{\Gamma((2l+1)\lambda-\lambda x/\pi)\Gamma(1+(2l-1)\lambda-\lambda x/\pi)}.$$

The Crossing Symmetry only make some constraints on the transformation factor $f(\theta)$ and $g(\theta)$. So we have obtained the reflection matrix $R_{\mathrm{T}}(\theta)$ for Thirring Model with the boundary (14),(15) and (19):

$$R_{\mathrm{T}}(\theta) = R(\lambda\theta) \tag{35}$$

which satisfy the boundary Y-B equation andthe C-U condition. Now we have to compare this $R_{\rm T}$ matrix with $R_{\rm S}$ of [1] using the invariant quantities[5] to find if the Thirring model with this boundary is equivalent to the Sine-Gordon model.

Indeed, the invariants satisfy the following relations:

$$P^{\pm}_{\rm T}(\theta)P^{\pm}_{\rm T}(-\theta) = \cos(\xi + i\lambda\theta)\cos(\xi - i\lambda\theta)R_s(\theta)R_s(-\theta) \tag{36}$$

$$Q^{\pm}_{\rm T}(\theta)Q^{\mp}_{\rm T}(-\theta) = \frac{1}{4}k^2\sinh^2(2\lambda\theta)R_s(\theta)R_s(-\theta), \tag{37}$$

where ξ and k are two formal parameters involved in the results of [1], and

$$R_s(\theta)R_s(-\theta) = [\cos^2(\xi) + \sinh^2(\lambda\theta) + k^2\sinh^2(\lambda\theta)\cosh^2(\lambda\theta)]^{-1}. \tag{38}$$

Inserting eq(27) into eq(36) and eq(37), one can get the relation between the boundary parameters (h, ϕ_0) and the formal parameters (ξ, k) used in [1]. Thus,

$$\begin{aligned}&[M\cosh(\lambda\theta) + h\cosh(\lambda\theta + i\beta\phi_0)][M\cosh(\lambda\theta) + h\cosh(\lambda\theta - i\beta\phi_0)]/(P_{\rm T}(\theta)P_{\rm T}(-\theta))\\ &= \cos(\xi - i\lambda\theta)\cos(\xi + i\lambda\theta)[\cos^2(\xi) + \sinh^2(\lambda\theta) + k^2\sinh^2(\lambda\theta)\cosh^2(\lambda\theta)]^{-1}\end{aligned} \tag{39}$$

[5] It is difficult to compare directly the R matrices, because there is an ambiguity on both sides, and the results given in [1] involve infinite products of Γ-functions as follows:

$$P^{\pm}(\theta) = \cos(\xi \pm \lambda u)R_s(u), \qquad Q^{\pm}(\theta) = \frac{k}{2}\sin(2\lambda u)R_s(u),$$

where $u = -i\theta$, $R_s(u) = R_0(u)R_1(u)$, $\quad R_0(u) = \frac{F_0(u)}{F_0(-u)}$,

$$F_0(u) = \frac{\Gamma(1-2\lambda u/\pi)}{\Gamma(\lambda - 2\lambda u/\pi)} \times \Pi_{k=1}^{\infty}\frac{\Gamma(4\lambda k - 2\lambda u/\pi)\Gamma(1+4\lambda k - 2\lambda u/\pi)\Gamma(\lambda(4k+1))\Gamma(1+\lambda(4k-1))}{\Gamma(\lambda(4k+1) - 2\lambda u/\pi)\Gamma(1+\lambda(4k-1) - 2\lambda u/\pi)\Gamma(1+4\lambda k)\Gamma(4\lambda k)},$$

$R_1(u) = \frac{1}{\cos\xi}\sigma(\eta, u)\sigma(i\vartheta, u)$, where

$$\sigma(x, u) = \frac{\Pi(x, \pi/2 - u)\Pi(-x, \pi/2 - u)\Pi(x, -\pi/2 + u)\Pi(-x, -\pi/2 + u)}{\Pi^2(x, \pi/2)\Pi^2(-x, \pi/2)}$$

$$\Pi(x, u) = \Pi_{l=0}^{\infty}\frac{\Gamma(1/2 + (2l+1/2)\lambda + x/\pi - \lambda u/\pi)\Gamma(1/2 + (2l+3/2)\lambda + x/\pi)}{\Gamma(1/2 + (2l+3/2)\lambda + x/\pi - \lambda u/\pi)\Gamma(1/2 + (2l+1/2)\lambda + x/\pi)}$$

$$\sigma(x, u)\sigma(x, -u) = [\cos(x + \lambda u)\cos(x - \lambda u)]^{-1}\cos^2 x$$

there parameters η and ϑ are determined through the equations

$$\cos\eta\cosh\vartheta = -\frac{1}{k}\cos\xi, \qquad \cos^2\eta + \cosh^2\vartheta = 1 + \frac{1}{k^2}.$$

and also,

$$
\begin{aligned}
&M^2\sinh^2(2\lambda\theta)/(P_{\mathrm{T}}(\theta)P_{\mathrm{T}}(-\theta))\\
&=k^2\sinh^2(2\lambda\theta)[\cos^2(\xi)+\sinh^2(\lambda\theta)+k^2\sinh^2(\lambda\theta)\cosh^2(\lambda\theta)]^{-1}.
\end{aligned}
\tag{40}
$$

For the fixed boundary case, $h\to\infty$, we get $k=0$ from eq(40), and eq(39) becomes an identity; it does not lead to any constraints on ξ. For the free boundary case, $h=0$, we get $\cos^2\xi=1$ and $k^2=1$ from eqs(39,40). It agrees with the results of[1] $k=[\sin(\frac{\lambda\pi}{2})]^{-1}$ only when $\lambda=2n+1, n=0,\pm1,\pm2,\ldots$. This k was introduced in [1] in order to get a pole of R matrix at $\theta=i\frac{\pi}{2}$ for free boundary. This pole can also be obtained by putting a transformation factor such as $f(\theta)=g(\theta)$ and $f(\theta)f^{\dagger}(-\theta)=\frac{s(h/M)+1-i\sinh\theta}{s(h/M)+1+i\sinh\theta}$, where s is an arbitrary function satisfing $s(h/M)\Big|_{h=0}=0$ (Such a 'CDD ambiguity ' $\Phi(\theta)$ will not change any of the Boundary Yang-Baxter equations, the Unitary condition, or Crossing symmetry) which means we can not determine the k only from the pole. However the $P^{\pm}(\theta)$ of [1] tends to zero, but our $P_{\mathrm{T}}^{\pm}(\theta)$ does not when $\lambda=$ even integer .

For the general case, we get

$$
\begin{cases}
k^2=\frac{M^2}{M^2+2hM\cos(\beta\phi_0)+h^2}\\
\sin^2(\xi)=\frac{h^2\sin^2(\beta\phi_0)}{M^2+2hM\cos(\beta\phi_0)+h^2}.
\end{cases}
\tag{41}
$$

It reproduces the result of [22] when $\lambda=1$.

Conclusion and Discussion

We have obtained the reflection matrix $R_{\mathrm{T}}(\theta)$ for the Thirring model with boundary (14), (15) and (19) which is regarded as the perturbation of the free boundary condition (15). This $R_{\mathrm{T}}(\theta)$ matrix is related directly to physical boundary parameters for the sine-Gordon model with integrable boundary (1) by using the relation between the sine-Gordon model and the Massive Thirring model. This R-matrix has a degree of freedom which is similar to the 'CDD ambiguity ', but its elements can be used to construct invariant quantities so that we can compare them with those of Ghoshal and Zamolodchikov as in eq(36) and eq(37). We found the simple boundary (14) for Thirring model is equivalent to boundary (1) for Sine-Gordon model at least provided the coupling constant β satisfies $\frac{8\pi}{\beta^2}-1=2n+1$.

It should be noted that the soliton $\psi(x)$ and $A(\theta)$ in (21) are both fermionic operators. However, the Zamolodchikovs soliton operators $A(\theta)$ are neither bosonic nor fermionic in the general case. Only when $\lambda = 2n+1$, can they be regarded as fermionic operators. This is just why our $R_{\mathrm{T}}(\theta)$, which satisfies the boundary Y-B equation and the C-U condition is equivalent to Ghoshal and Zamolodchikovs' only when $\lambda = 2n+1$. We conjecture one can get $R_{\mathrm{T}}(\theta)$ which is equivalent to Ghoshal and Zamolodchikovs' for all λ from equivalence between eq.(1) and eq.(14) when the $\psi(x)$ is expanded in terms of Zamolodchikov's soliton operators.

It is worth noting that while the boundary condition (2) is the unique form of integrable boundary condition for the sine-Gordon model, at least on the assumption its boundary potential $\mathcal{B}(\phi)$ only depends on ϕ, the boundary condition (20) is not the only integrable boundary condition for the massive Thirring model even if its boundary potential depends only on ψ_i (and not its x-derivatives) at $= 0$. Moreover, there is other possibility leading to $j^0(x,t)\Big|_{x=0} = 0$ besides (12), i.e. there are other free boundary conditions for the Thirring model. The simplest case is $\mathcal{L}_{\mathrm{T_{free}}} = \mathcal{L}_{\mathrm{T_{bulk}}}$, but it is trivial[6]. Another non-trivial free boundary condition is

$$
\begin{aligned}
&(1+icM)\psi_1(0) = a\psi_2(0) + b\psi_2^\dagger(0) - idM\psi_1^\dagger(0) \\
&-icg\left(\psi_1^\dagger(0)\psi_1(0^-) + \psi_1^\dagger(0^-)\psi_1(0)\right)\psi_2(0) - idg\left(\psi_1^\dagger(0)\psi_1(0^-) + \psi_1^\dagger(0^-)\psi_1(0)\right)\psi_2^\dagger(0).
\end{aligned}
$$

It is easy to check they lead to $j^0(x,t)\Big|_{x=0} = 0$ too, if

$$
\begin{cases}
|c|^2 - |d|^2 = 0 \\
|b|^2 - |a|^2 = 1 + iM(c - c^*) \\
ac^* - a^*c + bd^* - b^*d = 0.
\end{cases}
$$

In other words, the massive Thirring model with boundary is not equivalent completely to the sine-Gordon model with boundary. The integrable boundary condition for the massive Thirring model has more freedom than that for the sine-Gordon model. What is the general integrable condition of massive Thirring model? Obviously, there is much room for further development.

[6] There is a similar case in Ising model[1]. Recently, Mourad and Sasaki[37] also found that the solutions become trivial when they use $\mathcal{L}_{\mathrm{bulk}}$ as free boundary Lagrangian for nonlinear sigma model on half plane.

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References

[1] S. Ghoshal and A. Zamolodchikov, *Int. J. Mod. Phys.* **A9** (1994) 3841.

[2] E. Corrigan, P.E.Dorey, R.H.Rietdijk, R.Sasaki, *Phys. Lett.* **B333**(1994)83.

[3] P.Bowcock, E. Corrigan, P.E.Dorey, R.H.Rietdijk, *Nucl. Phys.* **B445**(1995)469-500.

[4] A.MacIntyre, *J. Phys. (Math. Gen.)***A28**(1995) 1089-1100.

[5] S. Ghoshal, *Int. J. Mod. Phys.* **A9** (1994) 4801.

[6] S.Skorik, H. Saleur, Boundary bound states and boundary bootstrap in the sine-Gordon model with Dirichlet boundary condition, hep-th/9502011.

[7] S.Penati, D.Zanon, *Phys. Lett.* **B358** (1995) 63-72.

[8] A. LeClair, Quantum Theory of Self-Induced Transparency, CLNS 95/1336, hep-th/9505086.

[9] X.G.Wen, *Phys. Rev.* **B44**(1991)5708.

[10] P.Fendley, F. Lesage, H.Saleur, *J. Stat. Phys.* **79** (1995) 799-819.

[11] J.Cardy, *Nucl. Phys.* **B324** (1989) 581.

[12] J.Cardy, D.Lewellen, *Phys. Lett.* **B259**(1991) 274.

[13] Ya. Arefeva and V.E. Korepin, *JETP Lett.* **20**(1974)312.

[14] A.B.Zamolodchikov and Al.B. Zamolodchikov, *Ann. Phys.* **120**(1979) 53 .

[15] V.E. Korepin, *Soviet J. Theor. Math. Phys.***41**(1979)953.

[16] P.Fendley, H.Saleur,*Nucl. Phys.* **B428**(1994)681.

[17] A.Fring, R.Köberle, *Nucl. Phys.* **B421** (1994) 159.

[18] R. Sasaki, *Interface between Physics and Mathematics*, eds, W. Nahm and J-M Shen, (world Scientific 1994) 201.

[19] F. Guinea, V. Hakin, A. Muramatsu, *Phys. Rev. Lett.* **54**(1985)263.

[20] C. Kane, M.Fisher, *Phys. Rev.* **B46**(1992) 15233.

[21] S.Skorik, H. Saleur and N.P.Warner, *Nucl. Phys.* **B441** (1995) 421-436.

[22] M.Ameduri, R. Konik and A. LeClair, *Phys. Lett.* **B354** (1995) 376-382.

[23] S. Coleman, *Phys. Rev.* **D 11**(1975) 2088-2097.

[24] S. Mandelstam, *Phys. Rev.* **D 11**(1975) 3026-3030.

[25] D.Bernard and A. LeClair, *Commun. Math. Phys.* **142** (1991) 99.

[26] S.J.Chang, R.Rajaraman, *Phys. Lett.* **B313** (1993)59.

[27] H.X. Yang and Y.X. Chen, *Phys. Lett.* **B337** (1994) 95 and 102.

[28] H.X.Yang, K.Li and Z.M.Sheng, *J. Phys.* **A27** (1994)L677.

[29] S.Lukyanov, *Commun. Math. Phys.* **167** (1995) 183-226.

[30] B.Y.Hou, K.J.Shi, Y.S.Wang, W.L.Yang, L.Chao, Bosonic Realization of Boundary Operators in $SU(2)$-invariant Thirring Model, hep-th/9503178.

[31] D.J.Kaup, A.C.Newell, *Lett. Nuovo. Cimento* **20** (1977) 325-332.

[32] T.R. Klassen, E.Melzer, *Int. J. Mod. Phys.* **A8**(1993)4131.

[33] C.M. Sommerfield, *Ann. Phys.* **26** (1963)1-43.

[34] A.V.Mikhailov, *JETP Lett.* **23** (1976) 320.

[35] Y.Z.Zhou, K.D.Schotle, *Phys. Rev.* **D47** (1993) 1281

[36] Al.B.Zamolodchikov, *Inter J. Mod. Phys.* **A10** (1995)1125-1150.

[37] M.F. Mourad, R. Sasaki, Non-linear Sigma Models on a Half Plane, YITP-95-4, hep-th/9509153.